VAGUE - Near Rings, Near Fields and Boolean Rings

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Abstract

In this paper, we introduce Vague – Near Rings and Vague - Near Fields, Vague Near Fields, Boolean Rings

Keywords: Vague set, Vague Additive group, Vague Rings, Vague Field, Vague Near Rings, Vague Near Field.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [6] since then this idea has been applied to other algebraic structures such as groups, Rings Fields etc., with the development of fuzzy set it is widely used in many Fields mean while the deficiency of fuzzy sets are also attracting the attention such as fuzzy set is single function it cannot express the evidence of supporting and opposing. Based on this reason the concept of Vague set [3] introduced by Gau in 1993.

Vague sets as an extension of fuzzy sets. The idea of Vague sets is that the membership of every element can be divided in to two aspects including supporting and opposing the notion of fuzzy groups defined by Rosen Field [4] is the first application fuzzy set theory on algebra. Ranjit Biswas [5] initiated the study of Vague algebra by studying Vague groups under multiplication.

We introduced the notions of Vague additive groups [1]. Vague Rings and Vague Fields [2]. As an extension of our mentioned work, we also introduce Vague Near Rings and Near Fields.

2. Preliminaries

In this section we given some definitions and state some results for later use.

Definition 2.1: A Vague set A in the universe of discourage X is characterized by two membership functions given by

1) A truth membership function $t_A : x \to [0, 1]$ and
2) A false membership function $f_A : x \to [0, 1]$

Where $t_A(x)$ *is* a lower bound of the grade of membership of x derived from the "evidence for x" and $f_A(x)$ is a lower bound on the "negation of x" derived from the "evidence for x" and $f_A(x)$ is a lower bound on the "negation of x" derived from the "evidence against x" and $t_A(x) + f_A(x) \leq 1$. Thus, the grade of membership of x in the

Vague set A is bounded by subinterval $[t_A(x), 1 - f_A(x)]$ of $[0,1]$. The Vague set A is written as $A = \{x, [t_A(x) f_A(x) / \ x \in X\}$ where the interval $[t_A(x), 1 - f_A(x)]$ is called the Vague value of x in the Vague set A and denoted by $V_A(x)$.

Example 2.1: $A = \{0 < 0.7, 0.2 >, \ 1 < 0.8, 0.1 >\}$ is a Vague set.

Definition 2.2: A Vague set A of a Universal set X with $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in X$ is called the zero Vague set of X.

Definition 2.3: A Vague set A of universal set X with $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in X$ is called the unit Vague set of X.

Definition 2.4: Let $(X, +)$ be a group. A Vague set A of X is called a Vague additive group (Briefly VAG) of x if the following conditions are satisfied.

1) $V_A(x + y) \leq' \max\{V_A(x), \ V_A(y)\}$ for all $x, y \in X$.
2) $V_A(-x) \leq V_A(x)$ for all $x \in X$.

Example 2.2 : Let $S = \{g_1, \ g_2, \ g_3 \ and \ g_4\} \ where \ g_1 = (0,0) \ g_2 = (1,0), \ g_3 = (0,1) \ and \ g_4 = (1,1)$ define addition modulus 2 on S as follows.

t_2	g_1	g_2	g_3	g_4
g_1	g_2	g_2	g_3	g_4
g_2	g_2	g_1	g_4	g_3
g_3	g_3	g_4	g_1	g_2
g_4	g_4	g_3	g_2	g_1

Clearly $(S +_2)$ is an additive abelian group.

Let A be a Vague set of S defined by $t_A(x) = \begin{cases} 0.6, if \ x = g_1 \\ 0.7, if \ x = g_2 \\ 0.8, if \ x = g_3, g_4 \end{cases}$ and

$$f_A(x) = \begin{cases} 0.3, if \ x = g_1 \\ 0.2, if \ x = g_2 \\ 0.1, if \ x = g_3, g_4 \end{cases}$$

Then, A is a Vague additive abelian group.

Definition 2.5: Let X be a Ring and R be a Vague set of X. Then R is called a Vague Ring of X if the following conditions are satisfied.

1. $V_R(x + y) \leq \max\{V_R(x), \ V_R(y)\} \ for \ all \ x, y \in X$.
2. $V_R(-x) \leq V_R(x) \ for \ all \ x \in X$.
3. $V_R(xy) \geq \min\{V_R(x), \ V_R(y)\} \ for \ all \ x, y \in X$.

Example 2.3: Let $R = \{0, 1\}$ and the addition modulo 2 and multiplication modulo 2 operations are defined as follows.

$+_2$	0	1
0	0	1
1	1	0

and

$\cdot_2$	0	1
0	0	0
1	0	1

Clearly, $(R, +_2, \cdot_2)$ is a Ring.

Let R be a Vague set of X defined by $t_R(x) = \begin{cases} 0.7, & if\ x = 0 \\ 0.8, if\ x = 1 \end{cases}$ and $f_R(x) = \begin{cases} 0.2, if\ x = 0 \\ 0.1, if\ x = 1 \end{cases}$

Then, R is a Vague Ring.

Definition 2.6: Let X be a Field and F be a Vague set of X. Then, F is called a Vague Field of X if the following conditions are satisfied.

1) $V_F\,(x + y) \leq max\ \{V_F(x),\ V_F\,(y)\}\ for\ all\ x,\ y \in X$
2) $V_F(-x) \leq V_F(x)\ for\ all\ x \in X$
3) $V_F\,(xy) \geq \min\ \{V_F(x),\ V_F\,(y)\}\ for\ all\ x, y \in X$
4) $V_F\,(x^{-1}) \geq V_F(x)\ for\ all\ x \in X$

Example 2.4: Let $F = \{0, 1, 2\}$ and the addition modulo 3 and multiplication modulo 3 operations are defined as follows.

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

$\cdot_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Clearly, $(F\ +_3\ ,\ \cdot_3)$ is a Field.

Let F be a Vague set of X defined by $t_F(x) = \begin{cases} 0.7, if\ x = 0 \\ 0.8, if\ x = 1,2 \end{cases}$ and $f_F(x) = \begin{cases} 0.2, if\ x = 0 \\ 0.1, if\ x = 1,2 \end{cases}$

Then, F is a Vague Field of X.

3. Vague Near Rings

Definition 3.1: A right Near Ring is a set N together with two binary operations "+" and "**.**" such that

(i) $(N, +)$ is a group (not necessarily Abelian)

(ii) $(N, .)$ is a semi group

(iii) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1,\ n_2, n_3 \in N$.

N is called a right Near Ring.

Addition Table

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Multiplication Table

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	C

Consider the set $N = \{0, a, b, c\}$ given by the explicit addition and multiplication in the above Tables. Here the additive group is the Klein's four group.

If it can be verified that, $(N, +, \ .)$ is a marbing but not a Ring. Observe that, $b.a + b.c = a + b = c$ where as $b.(a + c) = bb = b$.

$\therefore b.a + b.c \ = c \neq b \neq b.(a + c)$.

This shows that, N fails to satisfy the left distributive law. This is an example of a right Near Ring that is not a left Near Ring. Also, this provides an example of a Near Ring that is not a Ring.

Definition 3.2: Let N be a Near Ring and R be a Vague set of N. Then R is called a Vague Near Ring of N if the following conditions are satisfied.

1) $V_R(x + y) \leq \max\{V_R(x), V_R(y)\}\ for\ all\ x, y\ \in N$.
2) $V_R(-x) \leq\ V_R\ (x)\ for\ all\ x\ \in N$.
3) $V_R\ (xy) \leq \min\{V_R(x),\ V_R(y)\}\ for\ all\ x, y\ \in N$.

Example 3.3: By the above tables set of N defined let R be a Vague set of N defined by

$t_R(x) = \begin{cases} 0\ \ if\ \ x = 0 \\ 0.7\ if\ \ x = a \\ 0.8\ \ if\ \ x = b, c \end{cases}$ and $F_R(x) = \begin{cases} 1\ \ if\ \ x = 0 \\ 0.2\ if\ \ x\ = a \\ 0.1\ \ if\ \ x = b, c \end{cases}$ then, R is a Vague Near Ring

4.1. Near Field

Definition 4.1.1: A Near Field is an algebraic structure similar to a Division Ring except that it has only one of the two distributive laws. (or) A Near Field us a Near Ring in which there is a multiplicative identity and every non zero element has a multiplicative inverse.

Example 4.1.1: Any Division Ring is a Near is a Near Field (including any Field). Let $H_R = \mathbb{R}\ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^4$ be the 4 - dimensional real vector space. Elements of H_R are called real quarternions. In H_R define addition and multiplication as follows.

Addition: Addition is defined coordinate wise multiplication table for $\{\pm \mathbf{1}, \pm \boldsymbol{i}, \pm \boldsymbol{j}, \pm \boldsymbol{k}\}$

.	**1**	**i**	**j**	**k**	**-1**	**-i**	**-j**	**-k**
1	**1**	**i**	**j**	**k**	**-1**	**-i**	**-j**	**-k**
i	**i**	**-1**	**k**	**-j**	**-i**	**l**	**-k**	**j**
j	**j**	**-k**	**-1**	**i**	**-j**	**k**	**1**	**-i**
k	**k**	**j**	**-i**	**-1**	**-k**	**-j**	**i**	**l**
-1	**-1**	**-i**	**-j**	**-k**	**l**	**i**	**j**	**k**
-i	**-i**	**1**	**k**	**j**	**i**	**-1**	**-k**	**-j**
-j	**-j**	**k**	**1**	**i**	**j**	**-k**	**-1**	**-i**
-k	**-k**	**j**	**i**	**1**	**k**	**-j**	**-i**	**-1**

With respect to this addition and multiplication the real quaternions form a Ring which is not Commutative because $ij = k = -ji$

$1 = (1, 0, 0, 0)$ is the unity element and given $V = (x_1, x_2, x_3, x_4) \neq 0$ *we have* $\sum_{i=1}^{4} x_i^2 \neq 0$ and hence $V^{-1} = \frac{1}{\sum_{i=1}^{4} x_i^2}(x_1, -x_2, -x_3 - x_4) \in H_R$. Thus, H_R is a Division Ring which is not a Field.

$\therefore$ $H_{\mathbb{R}}$ is a Division Ring $\Longrightarrow H_{\mathbb{R}}$ is a Near Field.

4.2. Vague Near Field: Let X be a Near Field and F be a Vague set of X. Then, F is called a Vague Near Field of X if the following conditions are satisfied.

1) $V_F(x + y) \leq \max\{V_F(x), V_F(y)\}$ *for all* $x, y \in X$
2) $V_F(-x) \leq V_F(x)$ *for all* $x \in X$.
3) $V_F(xy) \leq \min\{V_F(x), V_F(y)\}$ *for all* $x, y \in X$.
4) $V_F(x^{-1}) \geq V_F(x)$ *for all* $x \in X$.

Example 4.2.1: In the above example $t_A(x) = \begin{cases} 0 \ \ if \ \ x = \pm 1 \\ 0.1 \ \ if \ \ x = \pm i \\ 0.2 \ if \ x = \pm j, \pm k \end{cases}$

$$f_A(x) = \begin{cases} 1 \ if \ x = \pm 1 \\ 0.8 \ if \ x = \pm i \\ 0.7 \ if \ x = \pm j, \pm k \end{cases}$$

Then, F is a Vague Near Field.

Example 4.2.2: In this case, we take the set of alphabets $F = \{x, a, b, c, d, e, f\}$ to test with designed Caley tables[7].

Addition Table:

+	**x**	**a**	**b**	**c**	**d**	**e**	**f**
x	x	a	b	c	d	e	f
a	a	b	c	d	e	f	x
b	b	c	d	e	f	x	a
c	c	d	e	f	x	a	b
d	d	e	f	x	a	b	c
e	e	f	x	a	b	c	d
f	f	x	a	b	c	d	e

Multiplication Table:

.	**x**	**a**	**b**	**c**	**d**	**e**	**f**
x	x	x	x	x	x	x	x
a	x	a	b	c	d	e	f
b	x	b	c	a	e	f	d
c	x	c	a	b	f	d	e
d	x	d	f	e	a	c	b
e	x	e	d	f	b	a	c
f	x	f	e	a	c	b	a

The above set is a Ring and Division Ring but not a Commutative Ring.

The above example is a Near Field. Now we define a Vague Near Field by giving Vague values for the above set $F = \{x, a, b, c, d, e, f\}$ by

$$t_F(x) = \begin{cases} 0, if\ x = x \\ 0.1, if\ x = a, d \\ 0.2, if\ x = b, c, e, f \end{cases} \text{ and } f_F(x) = \begin{cases} 1, if\ x = x \\ 0.8, if\ x = a, d \\ 0.7, if\ x = b, c, e, f \end{cases}$$

Then, F is a Vague Near Field.

5.1. Boolean Ring: A Ring R in which every element is an idempotent is called a Boolean Ring (i.e., $\forall a \in R, a^2 = a$).

Example 5.1: Let $X = \{a, b, c\}\ and\ P(X)$ be the power set of X i.e., set of all subsets of X. Define addition and multiplication on P(X) as follows.

Addition is the symmetric difference of sets, namely

$$A + B = A\,\Delta B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$$

Multiplication is the intersection of sets i.e., $AB = A \cap B$.

Under these operations $P(X)$ is a Commutative Ring with unity. This is a Boolean Ring.

Addition Table

$P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$

$$= \{\emptyset, A, B, C, P, Q, R, X\}$$

+	∅	A	B	C	P	Q	R	X
∅	∅	A	B	C	P	Q	R	X
A	A	∅	P	R	B	X	C	Q
B	B	P	∅	Q	A	C	X	R
C	C	R	Q	∅	X	B	A	P
P	P	B	A	X	∅	R	Q	C
Q	Q	X	C	B	R	∅	P	A
R	R	C	X	A	Q	P	∅	B
X	X	Q	R	P	C	A	B	∅

From the table, it is clear that, additive identity is $\emptyset$ and every set is its own inverse.

Multiplication Table

.	∅	A	B	C	P	Q	R	X
∅	∅	∅	∅	∅	∅	∅	∅	∅
A	∅	A	∅	∅	A	∅	A	A
B	∅	∅	B	∅	B	B	∅	B
C	∅	∅	∅	C	∅	C	C	C
P	∅	A	B	∅	P	B	A	P
Q	∅	∅	B	C	B	Q	C	Q
R	∅	A	∅	C	A	C	R	R
X	∅	A	B	C	P	Q	R	X

From the table, it is clear that, additive identity is X.

Every element is idempotent i.e., $A.A = A \implies A^2 = A$.

$\therefore [P(X), +\ .]$ is a Boolean Ring.

In the above Boolean Ring $P(X)$. if R is a. Vague set of $P(X)$. The Vague set $R = \{ x, [t_R(x), 1 - f_R(x)]$ on R.

Defined by $t_R(x) = \begin{cases} 0, if\ x = \phi \\ 0.1, if\ x = A \\ 0.2, if\ x = B \\ 0.3, if\ x = C, P, Q, R, X \end{cases}$ and $1 - f_R(x) = \begin{cases} 0, if\ x = \phi \\ 0.2, if\ x = A \\ 0.3, if\ x = B \\ 0.4, if\ x = C, P, Q, R, X \end{cases}$ is a Vague Boolean Ring

5.2. Vague Boolean Ring: (1) $V_R\ (A - B) \leq max\ \{V_R(A), V_R(B)\}$

(2) $V_R(A) \leq\ V_R(-A)$

(3) $V_R(A.B) \leq \min\{V_R A,\ V_R(B)\}$

6.1. Direct Product Ring:

Let R and S be two Rings. Let $T = R\ \times S$ be the Cartesian product of R and S. Define addition and multiplication on T coordinate - wise as follows

$(a, x) +\ (b, y\,) =\ (a + b,\ \ x + y)\ and$

$(a, x\,)\,(b, y) =\ (ab,\ xy)$ under these operations T is a Ring called the direct product (or) Cartesian product of R and S with $O_T = (O_R\ \ O_S)$

Example 6.1.1: Let $R = \{0,\ 1\} = Z_2\ and$

$S = \{0,\ \ 1\} = Z_2$

$T = R\ \times S = \{(0,0), (0,1)\ (1,0)\ (1,1)\}$

Addition Table

+	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 0)	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 1)	(0, 1)	(0, 0)	(1, 1)	(1, 0)
(1, 0)	(1, 0)	(1, 1)	(0, 0)	(0, 1)
(1, 1)	(1, 1)	(1, 0)	(0, 1)	(0, 0)

From the table

$O_T =\ (O_R, O_S)$

Multiplication Table

.	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(0, 1)	(0, 0)	(0, 1)	(0, 0)	(0, 1)
(1, 0)	(0, 0)	(0, 0)	(1, 0)	(1, 0)
(1, 1)	(0, 0)	(0, 1)	(1, 0)	(1, 1)

From the table

$l_T = (1_R\ \ 1_S\,)$

Every element has its own Inverse.

6.2. Vague Direct Product Ring: Let T be a Vague set of $R\ \times S$. Defined by

$$t_T(x) = \begin{cases} 0\ \ if\ \ x = (0,0) \\ 0.7\ if\ \ x = (0,1) \\ 0.8\ \ if\ x = (1,0), (1,1) \end{cases} \qquad f_T(x) = \begin{cases} 1\ \ if\ x = (0,0) \\ 0.2\ \ if\ x = (0,1) \\ 0.1\ \ if\ x = (1,0), (1,1) \end{cases}$$

Then, T is a Vague direct product Ring.

7. Conclusion: In this paper, the concept of Vague Near Ring, Vague Near Fields and Vague Boolean Rings are introduced. It is hoped that, these concepts will rise to the notions like Vague Division Rings and Vague polynomial Rings etc.,

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