Vague Topological Modules and Vague Topological Vector Spaces

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Abstract: In this paper we introduce the concepts of vague topological ring, vague topological field, vague topological module and vague topological vector space. We investigate some of their properties.

Keywords: Vague topological space, vague rings and vague fields.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [14] in 1965. The idea of fuzzy set is welcome because it handles uncertainty and vagueness which cantorian set could not address. The membership of an element to a fuzzy set is a single value between zero and one. The theory of fuzzy topology was introduced by C.L.Chang [5] in 1967. Several researches were conducted on the generalisations of the notions of fuzzy sets and fuzzy topology. The theory of vague sets was proposed by Gaw and Buchere [7] as an extension of fuzzy set theory and vague sets are regarded as a special case of content – dependent fuzzy sets. The idea of vague sets is that the membership of every element can be divided into two aspects including supporting and opposing. The theory of vague topology was introduced by Mariapresenti.L and Arockia Rani.I [11].

Now, we introduce the concepts of vague topological rings, vague topological modules, vague topological fields and vague topological vector spaces.

2. Preliminaries

Definition 2.1: [7] A vague set A in the universal of discourse X is characterized by two membership functions given by: A truth membership function $t_A: X \to [0,1]$ and A false membership function $f_A: X \to [0,1]$, where $t_A(x)$ is a lower bound of the grade of membership of x derived from the "evidence for x", and $f_A(x)$ is a lower bound on the negation of x derived from the "evidence against x" and $t_A(x) + f_A(x) \leq 1$.Thus, the grade of membership of x in the vague set A is bounded by subinterval $[t_A(x), 1 - f_A(x)]$ of $[0,1]$. This indicates that, if the actual grade of membership of x is $\mu(x)$ then, $t_A(x) \leq \mu(x) \leq 1 - f_A(x)$. Now, the vague set A is written as $A = \{(x, [t_A(x), 1 - f_A(x)])/x \in X\}$, where the interval $[t_A(x), 1 - f_A(x)]$ is called the value of x in the vague set A and denoted by $V_A(x)$.

Definition 2.2: [7] Let A and B be two vague sets of the form $A = \{(x, [t_A(x), 1 - f_A(x)])\}$ and $B = \{(x, [t_B(x), 1 - f_B(x)])\}$ then,

1. $A \subseteq B$ if and only if $t_A(x)\,) \leq t_B(x)$ and $1 - f_A(x) \leq 1 - f_B(x)$ for all $x \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $A^c = \{(x, [f_A(x), 1 - t_A(x)])\}$.
4. $A \cap B = \{min[\, t_A(x)\,), t_B(x)]\,, min[1 - f_A(x),\ 1 - f_B(x)]\}$.
5. $A \cup B = \{max[\, t_A(x), t_B(x)]\,, max[1 - f_A(x)\,, 1 - f_B(x)]\}$.

Definition 2.3: [11] A Vague Topology (VT) on X is a family T of Vague Sets (VS) in X satisfying the following axioms

1. $0, 1 \in T$.
2. $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$
3. $\cup\, G_i \in T$ for any family $\{\, G_i / i \in N\} \subseteq T$

In this case the pair (X, T) is called a Vague Topological Space (VTS) and any Vague Set in T is known as a Vague Open Set (VOS) in X. The complement A^c of a VOS A in a VTS (X, T) is called a Vague Closed Set (VCS) in X.

Definition 2.4: [3] An abelian group A is called a Topological Additive Group if a Topology is defined on the set A and following conditions are satisfied:

1. The mapping $(a, b) \to a + b$ of the Topological Space $A \times A$ on to the Topological space A is Continuous.
2. Additive inversion continuity condition: The mapping $a \to (-a)$ of the topological space A onto itself is continuous.

Remark: Let A be an abelian group then A is a Topological abelian group in the discrete or in the indiscrete topology.

Example 2.5: Let $G = \{0, 1\}$ and let $T = \{\, \Phi, \{0\}, \{1\}, G\}$ is a discrete topology on G. Let $A = \{0\}, B = \{1\}$ and define $A + B = (A - B) \cup (B - A)$

+	Φ	A	B	G
Φ	Φ	A	B	G
A	A	Φ	G	B
B	B	G	Φ	A
G	G	B	A	Φ

Table 1

So, $(G, T, +)$ is a topological additive group.

Definition 2.6: A ring R is called a topological ring if a topology is defined on the set R and the additive group of the ring R is a topological group in this topology and the following condition is satisfied:

The mapping $(a,b) \to a.b$ of the topological space $R \times R$ to the topological space R is continuous.

Example 2.7: Let $X = \{0,1,2\} = Z_3$ and let $T = \{\Phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, X\}$ is a discrete topology on X. We denote $A = \{0\}$, $B = \{1\}$, $C = \{2\}$, $P = \{0,1\}$, $Q = \{1,2\}$ and $R = \{0,2\}$ and define $A + B = (A - B) \cup (B - A)$ and $A.B = A \cap B$.

Addition Table:

$+$	Φ	A	B	C	P	Q	R	X
Φ	Φ	A	B	C	P	Q	R	X
A	A	Φ	P	R	B	X	C	Q
B	B	P	Φ	Q	A	C	X	R
C	C	R	Q	Φ	X	B	A	P
P	P	B	A	X	Φ	R	Q	C
Q	Q	X	C	B	R	Φ	P	A
R	R	C	X	A	Q	P	Φ	B
X	X	Q	R	P	C	A	B	Φ

Table 2

From the table 2 it is clear that, additive identity is Φ and every set is its own inverse.

Multiplication Table:

$.$	Φ	A	B	C	P	Q	R	X
Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ
A	Φ	A	Φ	Φ	A	Φ	A	A
B	Φ	Φ	B	Φ	B	B	Φ	B
C	Φ	Φ	Φ	C	Φ	C	C	C
P	Φ	A	B	Φ	P	B	A	P
Q	Φ	Φ	B	C	B	Q	C	Q
R	Φ	A	Φ	C	A	C	R	R
X	Φ	A	B	C	P	Q	R	X

Table 3

From the table 3 it is clear that, additive identity is X.

Intersection satisfies associative law.

Let $A, B, C \in T$, then

$$\mathrm{A} \cap (\mathrm{B} + \mathrm{C}) = \mathrm{A} \cap [(\mathrm{B} - \mathrm{C}) \cup (\mathrm{C} - \mathrm{B})]$$
$$= [\mathrm{A} \cap (\mathrm{B} - \mathrm{C})] \cup [\mathrm{A} \cap (\mathrm{C} - \mathrm{B})]$$
$$= [\mathrm{A} \cap \mathrm{B} \cap (\mathrm{X} - \mathrm{C})] \cup [\mathrm{A} \cap \mathrm{C} \cap (\mathrm{X} - \mathrm{B})].$$

$$(\mathrm{A} \cap \mathrm{B}) + (\mathrm{A} \cap \mathrm{C}) = [(\mathrm{A} \cap \mathrm{B}) - (\mathrm{A} \cap \mathrm{C})] \cup [(\mathrm{A} \cap \mathrm{C}) - (\mathrm{A} \cap \mathrm{B})]$$
$$= [(\mathrm{A} \cap \mathrm{B}) - \mathrm{C})] \cup [(\mathrm{A} \cap \mathrm{C}) - \mathrm{B})]$$
$$= [\mathrm{A} \cap \mathrm{B} \cap (\mathrm{X} - \mathrm{C})] \cup [\mathrm{A} \cap \mathrm{C} \cap (\mathrm{X} - \mathrm{B})].$$

So $\mathrm{A} \cap (\mathrm{B} + \mathrm{C}) = (\mathrm{A} \cap \mathrm{B}) + (\mathrm{A} \cap \mathrm{C})$.

Therefore $(X, T, +, .)$ is a topological ring.

Definition 2.8: A topological module is a module over a topological ring such that scalar multiplication and addition are continuous.

Example 2.9: Let $X = \{0, 1, 2\} = Z_3$ be a ring then $M = T = \{\Phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, X\}$ is a discrete topology on X. Clearly M is a topological $R -$ module.

Definition 2.10: Let X be an indiscrete space. The only continuous maps are constant maps from $X \rightarrow R$ and hence the ring R can be identified with the field R.

Example 2.11: Let X be an arbitrary set and $T = \{\Phi, X\}$ be an indiscrete topology on X. It can be observed from the following tables $(T, +, .)$ is a topological field.

Define $A + B = (A - B) \cup (B - A)$ and $A . B = A \cap B$

$+$	Φ	X
Φ	Φ	X
X	X	Φ

Table4

$.$	Φ	X
Φ	Φ	Φ
X	Φ	X

Table 5

Example 2.12: Let $X = Z_2 = \{0,1\}$ be a Galoi's field under addition modulo 2 and multiplication modulo 2 operations are as follows:

$+_2$	0	1
0	0	1
1	1	0

Table 6

$\cdot_2$	0	1
0	0	0
1	0	1

Table 7

Clearly $(X, +_2, \cdot_2)$ is a field.

If we take $T = \{\Phi, X\}$ be an indiscrete topology on X then $(T, +_2, \cdot_2)$ is a topological field.

Definition 2.13: A linear vector space X with a topology T on X is called a topological vector space if addition is a continuous function from $X \times X$ into X and multiplication by scalars is a continuous function from $F \times X$ into X.

Example 2.14: Let $X = Z_2 = \{0, 1\}$ be a vector space over the field $F = \{0, 1\} = Z_2$ and $T = \{\Phi, A, B, X\}$, where $A = \{0\}$, $B = \{1\}$, be the discrete topology on X. From the example 2.5, clearly$(T, +_2)$ ia an abelian additive topological group.

Suppose $0, 1 \in Z_2 = F$, then clearly

$$0(\Phi) = 0, \quad 0(A) = 0, \quad 0(B) = 0, \quad 0(X) = 0,$$
$$1(\Phi) = \Phi, \quad 1(A) = A, \quad 1(B) = B, \quad 1(X) = X.$$

Therefore T is a topological vector space.

Example 2.15: Let $X = Z_3 = \{0, 1, 2\}$ be a vector space over the field $= \{0, 1, 2\} = Z_3$. Then the set T in the example 2.7 is a topological vector space.

3. Vague Topological Rings

Definition3.1: A topological ring R is said to be a vague topological ring if it satisfies the following conditions:

1. $V_R(A \cup B) \leq max\{V_R(A), V_R(B)\}$
2. $V_R(A \cap B) \leq min\{V_R(A), V_R(B)\}$
3. $V_R(A + B) \leq max\{V_R(A), V_R(B)\}$
4. $V_R(AB) \leq min\{V_R(A), V_R(B)\}$.

Example 3.2: Let R be any ring and endow R with the indiscrete topology. Then, $T = \{\Phi, X\}$ is a topological ring.

Clearly, the vague set $A = \{\Phi < 0, 0 >, X < 1, 1 >\}$ is a vague topological ring.

Example3.3: Let $X = \{0, 1\}$, $T = \{\Phi, \{0\}, \{1\}, X\} = \{\Phi, A, B, X\}$. Then $(T, +, .)$ is a topological ring. The vague set R on X, $R = \langle \Phi, [0, 0.2]\rangle\langle A, [0.1, 0.2]\rangle, \langle B, [0.2, 0.2]\rangle, \langle X, [0.2, 0.2]\rangle\}$ is a vague topological ring.

Example 3.4: Let $X = \{0, 1, 2\}$ and $T = \{\Phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, X\}$

$$= \{\Phi, A, B, C, P, Q, R, X\}.$$

Then, $(T, +_3, ._3)$ is a topological ring.

The vague set $R = \langle x, [t_R(x), 1 - f_R(x)] \rangle$ on X defined by

$$\begin{aligned} t_R(x) &= 0 \; if \;\; x = \Phi \\ &= 0.1 \; if \; x = A \\ &= 0.2 \; if \; x = B \\ &= 0.3 \; if \; x = C, P, Q, R, X \end{aligned}$$

$$\begin{aligned} 1 - f_R(x) &= 0 \; if \;\; x = \Phi \\ &= 0.2 \; if \; x = A \\ &= 0.3 \; if \; x = B \\ &= 0.4 \; if \; x = C, P, Q, R, X \end{aligned}$$

is a vague topological ring.

Lemma 3.5: If R is a vague topological ring of a ring X then for all $\in X$, we have $V_R(-A) = V_R(A)$.

Theorem 3.6: Let R be a vague set of a ring X then R is a vague topological ring of a ring X if and only if R satisfies the following conditions:

1. $V_R(A - B) \leq max \{V_R(A), V_R(B)\}$
2. $V_R(A) \leq V_R(-A)$
3. $V_R(A \cap B) \leq min \{V_R(A), V_R(B)\}$.

Proof: Let R be a vague topological ring of a ring X. Then we have

$$\begin{aligned} t_R(A - B) &\leq max \{t_R(A), t_R(-B)\} \\ &= max \{t_R(A), t_R(B)\} \end{aligned}$$

Similarly, we can prove that, $1 - f_R(A - B) \leq max \{1 - f_R(A), 1 - f_R(B)\}$. It follows that, $V_R(A - B) \leq max \{V_R(A), V_R(B)\}$.

And by the definition, we get $V_R(A) \leq V_R(-A)$, $V_R(A \cap B) \leq min \{V_R(A), V_R(B)\}$.

Conversely suppose that, R is a vague set of a ring X. Also we have $V_R(A) \leq V_R(-A)$.

Then clearly $t_R(A + B) = t_R\left(A - (-B)\right)$

$$\leq max \{ t_R(A), t_R(-B)\}$$

$= max\{t_R(A), t_R(B)\}$

Similarly, we can prove that $1 - f_R(A + B) \leq max\{1 - f_R(A), 1 - f_R(B)\}$. It follows that, $V_R(A + B) \leq max\{V_R(A), V_R(B)\}$.

Hence, R is a vague topological ring of a ring X.

Theorem 3.7: Let R and S are vague topological rings of a ring X then $R \cap S$ is a vague topological ring of X.

Proof: Let R and S are vague topological rings of a ring X.

Then we have, $t_{R\cap S}(A - B) = min\{t_R(A - B), t_S(A - B)\}$

$\leq min\{max\{t_R(A), t_R(B)\}, max\{t_S(A), t_S(B)\}\}$

$= max\{min\{t_R(A), t_S(A)\}, min\{t_R(B), t_S(B)\}\}$

$= \max\{t_{R\cap S}(A), t_{R\cap S}(B)\}$

Similarly, we can prove that, $1 - f_{R\cap S}(A - B) \leq imax\{1 - f_{R\cap S}(A), 1 - f_{R\cap S}(B)\}$. It follows that, $V_{R\cap S}(A - B) \leq imax\{V_{R\cap S}(A), V_{R\cap S}(B)\}$.

Clearly, $V_R(A) \leq V_R(-A)$.

And $t_{R\cap S}(AB) = min\{t_R(AB), t_S(AB)\}$

$\leq min\{max\{t_R(A), t_R(B)\}, max\{t_S(A), t_S(B)\}\}$

$= max\{min\{t_R(A), t_S(A)\}, min\{t_R(B), t_S(B)\}\}$

$= max\{t_{R\cap S}(A), t_{R\cap S}(B)\}$

Similarly, we can prove that $1 - f_{R\cap S}(AB) \leq imax\{1 - f_{R\cap S}(A), 1 - f_{R\cap S}(B)\}$. It follows that, $V_{R\cap S}(AB) \leq imax\{V_{R\cap S}(A), V_{R\cap S}(B)\}$. Therefore, $R \cap S$ is a vague topological ring.

Definition 3.8: Let f be a mapping from a set X into a set Y. Let B be a vague set in Y. Then the inverse image of B, $f^{-1}(B)$ is the vague set in X by $V_{f^{-1}(B)}(x) = V_B\big(f(x)\big)$ for all $x \in X$.

Definition 3.9: Let f be a mapping from a set X into set Y. Let A be a vague set in X. Then the image of A, $f(A)$ is the vague set in Y by $V_{f(A)}(y) = isup\{V_A(z)/z \in f^{-1}(y)\}$, if $f^{-1}(y) \neq \emptyset$

$= [0, 0]$, otherwise.

Theorem 3.10: Let R and S are vague topological rings and f be a homomorphism from R into S. Let B be a vague topological ring of S, then the inverse image $f^{-1}(B)$ is a vague topological ring of R.

Proof: Let R and S are vague topological rings.

Then for all $C, D \in R$, we have $V_{f^{-1}(B)}(C - D) = V_B(f(C - D))$

$= V_B(f(C) - f(D))$

$\leq max\{V_A(f(C)), V_A(f(D))\}$

$$= max\,\{V_{f^{-1}(B)}(C), V_{f^{-1}(B)}(D)\}$$

Similarly, we can prove that, $V_{f^{-1}(B)}(CD) \leq max\{V_{f^{-1}(B)}(C), V_{f^{-1}(B)}(D)\}$. Hence, $f^{-1}(B)$ is a vague topological ring of R.

Theorem 3.11: Let R and S be vague topological rings and f be a homomorphism from R into S. Let A be a vague topological ring of R, then the image $f(A)$ is a vague topological ring of S.

Proof: Let R and S be vague topological rings. Let $U, V \in S$ such that

$V_{f(R)}(U) = isup\,\{V_R(U)\,/\,C \in f^{-1}(U)\}$ and $V_{f(R)}(V) = isup\,\{V_R(V)\,/\,D \in f^{-1}(V)\}$.

Then, $V_{f(R)}(U - V) = sup\{V_R(W)\,/\,W \in f^{-1}(U - V)\}$

$$\leq max\{V_R(C), V_R(D)\}$$
$$= imax\,\{V_{f(A)}(u), V_{f(A)}(v)\}.$$

Thus the image $f(A)$ of A is a vague topological ring of S.

4. Vague Topological Modules

Definition 4.1: Let R be a ring and M be a topological R- module. Then the set S of M is a vague topological module if it satisfies the following conditions:

1. $V_s(A + B) \leq max\,\{V_s(A), V_s(B)\}$ for all $A, B \in M$
2. $V_s(rA) \leq V_s(A)\ for\ all\ A \in M\ and\ for\ all\ r \in R$
3. $V_s(\Phi) = (0, 0)$.

Example 4.2: Let $(T, M, +)$ be a topological module in the example 2.7.

The vague set $S = \langle x, [t_s(x), 1 - f_s(x)] \rangle$ on X defined by

$$t_s(x) = 0\ if\ x = \Phi$$
$$= 0.2\ if\ x = A$$
$$= 0.3\ if\ x = B$$
$$= 0.4\ if\ x = C, P, Q, R, X$$
$$1 - f_s(x) = 0\ if\ x = \Phi$$
$$= 0.3\ if\ x = A$$
$$= 0.4\ if\ x = B$$
$$= 0.5\ if\ x = C, P, Q, R, X$$

is a vague topological module

Theorem 4.3: The intersection of a family of vague topological modules is also a vague topological module.

Proof: Let $\{M_i\,/\,i \in I\}$ be a family of vague topological modules and $M = \cap M_i$. Then

$$V_M(A + B) = \inf_{i\in I}\{V_{M_i}(A + B)\}$$

$$\leq \inf_{i\in I}\{\max\ (V_{M_i}(A), V_{M_i}(B))\}$$
$$= \max\{V_M(A), V_M(B)\}.$$

and

$$V_M(rA) = \sup_{i\in I}\{V_{M_i}(rA)\}$$
$$\leq \sup_{i\in I}\{\max V_{M_i}(A)\}$$
$$= \max\{\sup_{i\in I} V_M(A)\}$$
$$= V_M(A).$$

Hence, the intersection of a family of vague topological modules is also a vague topological module.

Theorem 4.4: Let M_1 and M_2 be two vague topological ring modules over the ring R and f be a linear transform of M_1 and M_2. Let M be a vague topological module of M_2, then the inverse image $f^{-1}(M)$ is a vague topological module of M_1.

Proof: Let $A, B \in M_1$ and $a, b \in R$, then

$$V_{f^{-1}(M)}(aA + bB) = V_M(f(aA + bB))$$
$$= V_M(af(A) + bf(B))$$
$$\leq \max\{V_M(f(A)), V_M(f(B))\}$$
$$= \max\{V_{f^{-1}(M)}(A), V_{f^{-1}(M)}(B)\}.$$

Hence, $f^{-1}(M)$ is a vague topological ring of M_1.

Theorem 4.5: Let M_1 and M_2 be two vague topological ring modules over the ring R and f be a linear transform of M_1 and M_2. Let M be a vague topological module of M_1, then the inverse image $f(M)$ is a vague topological module of M_2.

Proof: Let $\alpha, \beta \in M_2$. If either $f^{-1}(\alpha)$ or $f^{-1}(\beta)$ is empty then the inequality theorem $V_{f(M)}(aA + bB) \leq \max\{V_{f(M)}(A), V_{f(M)}(B)\}$ is satisfied for all $A, B \in M$ and for all $a, b \in R$.

Suppose that, neither $f^{-1}(\alpha)$ nor $f^{-1}(\beta)$ is empty.

Let $A_0 \in f^{-1}(\alpha), B_0 \in f^{-1}(\beta)$ then

$$V_M(A_0) = \sup_{A\in f^{-1}(\alpha)} V_M(A),$$
$$V_M(B_0) = \sup_{B\in f^{-1}(\beta)} V_M(B).$$

Then $V_{f(M)}(a\alpha + b\beta) = \sup_{r\in f^{-1}(a\alpha+b\beta)} V_M(r)$ where $r \in f^{-1}(A + B)$

$$\leq \max\{V_M(f(A_0)), V_M(f(B_0))\}$$
$$= \max\{V_{f(M)}(\alpha), V_{f(M)}(\beta)\}$$

Thus, the image $f(M)$ is a vague topological module of M_2.

5. Vague Topological Fields

Definition 5.1: A topological ring F is said to be a vague topological field if it satisfies the following conditions:

1. $V_F(A \cup B) \leq max\,\{V_F(A), V_F(B)\}$
2. $V_F(A \cap B) \leq min\,\{V_F(A), V_F(B)\}$
3. $V_F(A + B) \leq max\,\{V_F(A), V_F(B)\}$
4. $V_F(AB) \leq min\,\{V_F(A), V_F(B)\}$.

Example 5.2: Let X be any arbitary set and $T = \{\Phi, X\}$ be an indecrete topological space, then, $F = (T, +, .)$ is a topological field. Clearly, the vague set $A = \{\Phi < 0, 0 >, X < 1, 1 >\}$ is a vague topological field.

Example 5.3: Let $X = \{0, 1\}$ be a field, $T = \{\Phi, \{0\}, \{1\}, X\} = \{\Phi, A, B, X\}$. Then, $(T, +, .)$ is a topological field. The vague set R on X, $R = \langle \Phi, [0, 0.2] \rangle \langle A, [0.1, 0.2] \rangle, \langle B, [0.2, 0.2] \rangle, \langle X, [0.2, 0.2] \rangle\}$ is a vague topological field.

Lemma 5.4: If F is a vague topological field of a field X then

1. $A \in X$, we have $V_F(-A) = V_F(A)$,
2. $A \in X$, we have $V_F(A^{-1}) = V_F(A)$.

Proof: By definition, we have $V_F(-A) \leq V_F(A)$ for all $\in X$ .... (1)

Also, $V_F(A) = V_F(-(-A)) \leq V_F(-A)$ .... (2)

From (1) and (2), $V_F(-A) = V_F(A)$

Similarly, $V_F(A^{-1}) \leq V_F(A)$ for all $\in X$ .... (3)

Also, $V_F(A) = V_F((A^{-1})^{-1}) \leq (A^{-1})$ .... (4)

From (3) and (4), $V_F(-A) = V_F(A)$.

Theorem 5.5: Let F be a vague topological subset of a field X. Then F is a vague topological field of X if and only if F satisfies the following conditions:

1. $V_F(A - B) \leq max\,\{V_F(A), V_F(B)\}$ for all $A, B \in F$;
2. $V_F(A \cap B^{-1}) \leq min\,\{V_F(A), V_F(B)\}$ for all $A, B \in F$.

Proof: Let F be a vague topological subset of a field X. Then we have

$$V_F(A - B) \leq \max\{V_F(A), V_F(-B\}$$
$$= max\,\{V_F(A), V_F(B)\}.$$

Similarly, we can prove that, $V_F(A \cap B^{-1}) \leq min\,\{V_F(A), V_F(B)\}$ for all $A, B \in F$.

Theorem 5.6: The intersection of a family of vague topological fields is a vague topological field.

Proof: Let $\{F_i / \, i \in I\}$ be a family of vague topological fields and $F = \cap F_i$.

Then, $V_F(A-B) = {}^{min}_{i\in I}\left\{V_{F_i}(A-B)\right\}$

$\leq^{min}_{i\in I}\left\{max\{V_{F_i}(A), V_{F_i}(B)\}\right\}$

$= max\,\{V_F(A), V_F(B)\}$.

Similarly, we can prove that, $V_F(A \cap B^{-1}) \leq min\,\{V_F(A), V_F(B)\}$.

Hence, the intersection of a family of vague topological fields is also vague topological field.

6. Vague Topological Vector Space

Definition 6.1: Let S be a topological vector space over a field F. Then the set S is a vague topological vector space if it satisfies the following conditions:

1. $V_S(A + B)\, max\,\{V_S(A), V_S(B)\}$;
2. $V_S(aA) \leq V_S(A)$ for all $a \in F$;
3. $V_S(0) = (0, 0)$.

Example 6.2: Let (X, T) be a topological vector space in the example 2.15.The vague set $S = \{x, [t_S(x), 1 - f_s(x)]\}$ on X defined by

$$t_S(x) = 0\ if\ x = \Phi$$
$$= 0.3\ if\ x = A$$
$$= 0.4\ if\ x = B$$
$$= 0.5\ if\ x = C, P, Q, R, X$$
$$1 - f_S(x) = 0\ if\ x = \Phi$$
$$= 0.4\ if\ x = A$$
$$= 0.5\ if\ x = B$$
$$= 0.6 if\ x = C, P, Q, R, X$$

is a vague topological vector space.

Example 6.3: For a Vague Topological Vector space over the field Z_2:

$$X = Z_2 \times Z_2 = \{0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\}$$

$X = \{e, a, b, c\}$ where $e = (0,0), a = (0,1), b = (1,0), c = (1,1)$

$\tau = \{\Phi, \{e\}, \{a\}, \{b\}, \{c\}, \{e, a\}, \{e, b\}, \{e, c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{e, a, b\}, \{e, b, c\}, \{e, c, a\}, \{a, b, c\},$
$\{e, a, b, c\}\}$

For our convenience we take

$$A = \{e\}, B = \{a\}, C = \{b\}, D = \{c\}, E = \{e, a\}, F = \{e, b\}, G = \{e, c\}, P = \{a, b\}, Q = \{b, c\},$$
$$R = \{C, A\}, S = \{e, a, b\}, T = \{e, b, c\}, U = \{e, c, a\}, V = \{a, b, c\}, X = \{e, a, b, c\}$$

The Addition Table is as given below $A + B = (A - B)\ U\ (B - A)$

+	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X
Φ	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X
A	A	Φ	E	F	G	B	C	D	S	T	U	P	Q	R	X	V
B	B	E	Φ	P	R	A	S	U	C	V	D	F	X	G	Q	T
C	C	F	P	Φ	Q	S	A	T	B	D	V	E	G	X	R	U
D	D	G	R	Q	Φ	U	T	A	V	C	B	X	F	E	P	S
E	E	B	A	S	U	Φ	P	R	F	X	G	C	V	D	T	Q
F	F	C	S	A	T	P	Φ	Q	E	G	X	B	D	V	U	R
G	G	D	U	T	A	R	Q	Φ	X	F	E	V	C	B	S	P
P	P	S	C	B	V	F	E	X	Φ	R	Q	A	U	T	D	G
Q	Q	T	V	D	C	X	G	F	R	Φ	P	U	A	S	B	E
R	R	U	D	V	B	G	X	E	Q	P	Φ	T	S	A	C	F
S	S	P	F	E	X	C	B	V	A	U	T	Φ	R	Q	G	D
T	T	Q	X	G	F	V	D	C	U	A	S	R	Φ	P	E	B
U	U	R	G	X	E	D	V	B	T	S	A	Q	P	Φ	F	C
V	V	X	Q	R	P	T	U	S	D	B	C	G	E	F	Φ	A
X	X	V	T	U	S	Q	R	P	G	E	F	D	B	C	A	Φ

Additive identity elements is Φ

Every set has its own inverse.

Inverse of A is A only. Similarly, for the other elements also.

Multiplication Table is as given below $A.B = A \cap B$

.	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X
Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ
A	Φ	A	Φ	Φ	Φ	A	A	A	Φ	Φ	Φ	A	A	A	Φ	A
B	Φ	Φ	B	Φ	Φ	B	Φ	Φ	B	Φ	B	B	Φ	B	B	B
C	Φ	Φ	Φ	C	Φ	Φ	C	C	C	C	Φ	Φ	C	Φ	C	C
D	Φ	Φ	Φ	Φ	D	Φ	Φ	D	Φ	D	D	Φ	D	D	D	D
E	Φ	A	B	Φ	Φ	A	A	A	B	Φ	B	E	B	E	B	E
F	Φ	A	Φ	C	Φ	A	F	A	C	C	Φ	F	F	A	C	F
G	Φ	A	Φ	Φ	Φ	A	A	G	Φ	D	D	A	A	G	D	G
P	Φ	Φ	B	C	Φ	B	C	Φ	P	E	B	P	C	B	P	P
Q	Φ	Φ	Φ	C	D	Φ	C	D	C	Q	D	C	Q	D	Q	Q
R	Φ	Φ	B	Φ	D	B	Φ	D	B	D	R	B	Φ	R	R	R
S	Φ	A	B	C	Φ	E	F	A	P	Q	B	S	F	E	P	S
T	Φ	A	Φ	C	D	A	F	G	C	Q	D	F	T	G	Q	T
U	Φ	A	B	Φ	D	E	A	G	B	D	R	E	G	U	Q	U
V	Φ	Φ	B	C	D	B	C	D	P	Q	R	P	Q	R	V	V
X	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X

From the above two tables it is clear that, $(\tau +,.)$ is a Topological vector space. It is represented by V. Now, the vague set $S = (x, [t_S(x), f_S(x)])$ on V defined by

$$t_S(x) = 0 \; if \; x = \Phi$$
$$= 0.6 \; otherwise$$
$$1 - f_s(x) = 0 \; if \; x = \Phi$$
$$= 0.7 \; otherwise$$

Then S is a Vague Topological Vector space.

Example 6.4: Let $X = \frac{Z_2}{\langle 1+x+x^2 \rangle} = \{0, 1, \alpha, 1+\alpha\}$ be a set. Define addition modulo 2 and multiplication modulo 2 on X as follows:

Addition Table:

$+_2$	0	1	α	$1+\alpha$
0	0	1	α	$1+\alpha$
1	1	0	$1+\alpha$	α
α	α	$1+\alpha$	0	1
$1+\alpha$	$1+\alpha$	α	1	0

Multiplication Table:

$\times_2$	0	1	α	$1+\alpha$
0	0	0	0	0
1	0	1	α	1+α
α	0	1	1 + α	α
1 + α	0	1+α	1	α

Then $(X, +_2, \times_2)$ is a Field.

If we take $X = \{0, 1, \alpha, 1 + \alpha\}$

From the Example of Vague Topological Vector space, we can consider

$\tau = \{\Phi, \{0\}, \{1\}, \{\alpha\}, \{1+\alpha\}, \{0,1\}, \{0, \alpha\}, \{0,1+\alpha\}, \{1, \alpha\}, \{1,1+\alpha\}, \{\alpha,1+\alpha\}, \{0,1, \alpha\}, \{0,1,1+\alpha\}, \{0, \alpha,1+\alpha\}, \{1, \alpha,1+\alpha\}, \{0,1, \alpha,1+\alpha\}$

For our convenience we take

$\tau = \{\Phi, A, B, C, D, E, F, G, P, Q, R, S, T, U, V, X\}$,where $A = \{0\}$, $B = \{1\}$, $C = \{\alpha\}$, $D = \{1+\alpha\}$, $E = \{0,1\}$, $F = \{0, \alpha\}$, $G = \{0,1+\alpha\}$, $P = \{1, \alpha\}$, $Q = \{1,1+\alpha\}$, $R = \{\alpha, 1+\alpha\}$, $S = \{0,1,\alpha\}$, $T = \{0,1,1+\alpha\}$, $U = \{0, \alpha, 1+\alpha\}$, $V = \{1, \alpha, 1+\alpha\}$, $X = \{0,1,\alpha, 1+\alpha\}$.

We can write the tables for addition and multiplication tables as in the 6.3example of Topological vector space.

Hence, $(\tau, +, .)$ is a Topolocial Vector space.

We can define Vague Toplogical vector space as follows now the vague set $S = \{x, [t_S(x), f_S(x)]\}$ on τ defined by

$$t_S(x) = 0 \; if \; x = \Phi$$
$$= 0.5 \; otherwise$$
$$1 - f_S(x) = 0 \; if \; x = \Phi$$
$$= 0.6 \; otherwise$$

Theorem 6.5: If S is a vague topological vector space of a vector space X over a field F, then $V_S(\lambda A) = V_S(A)$ for all $0 \neq \lambda \in F$.

Proof: For all $A \in S$, we have $V_S(\lambda A) \leq V_S(A)$ for some $0 \neq \lambda \in F$.

Also, we have $V_S(A) = V_S(\lambda A \lambda^{-1}) \leq V_S(\lambda A)$. That is, $V_S(\lambda A) = V_S(A)$.

Theorem 6.6: Let X be a topological vector space over a field F and S is a vague set of X. Then S vague S is a vague topological vector space of X if and only if $V_S(aA + b\,B) \leq max\,\{V_S(A), V_S(B)\}$ for all $A, B \in S, a, b \in F$.

Proof: Suppose that S is a vague topological vector space of X. We have $V_S(aA) = V_S(A)$ and $V_S(bB) = V_S(B)$. Then we have $V_S(aA + b\,B) \leq max\,\{V_S(A), V_S(B)\}$. Conversely suppose that if $a = b = 1$, we have $V_S(A + B) \leq max\,\{V_S(A), V_S(B)\}$ and if $b = 0$ then $V_S(aA\,) \leq V_S(A)$. So, S is a vague topological vector space of X.

Theorem 6.7: Let R and S are vague topological vector spaces of vector space X over a field F. Then we have $R \cap S$ is a vague topological vector space of X.

Proof: Let R and S be vague topological vector spaces of vector space X over a field F.

Then we have, $V_{R\cap S}(A + B\,) = min\,\{V_R(A + B), V_S(A + B)\}$
$$\leq min\{max\{V_R(A), V_R(B)\}, max\{V_S(A), V_S(B)\}\}$$
$$= max\{min\{V_R(A), V_S(A)\}, min\,\{V_R(B), V_S(B)\}\}$$
$$= max\,\{V_{R\cap S}\,(A), \quad V_{R\cap S}\,(B)\}$$

and

$$V_{R\cap S}\,(\lambda A\,) = min\,\{V_R(\lambda A\,), V_S(\lambda A)\}$$
$$= min\,\{min[V_R(A\,), V_S(A)]\}$$

So $R\cap S$ is a vague topological vector space.

Conclusion:

We investigated vague topological rings, vague topological fields, vague topological modules and vague topological vector spaces. It is hoped that these concepts will rise to the notations like vague normed linear spaces, vague Hilbert spaces and vague Banach spaces etc.

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